Chapter 1

Distributions

The concept of distribution generalises and extends the concept of function.

A distribution is basically defined by its action on a set of test functions. It is indeed a linear functional $T$

$$T : \mathcal{F} \to \mathbb{C}$$

$$\phi(x) \to T(\phi).$$

(1.1)

from a space of functions $\mathcal{F}$ of real variable, called the test functions, to $\mathbb{C}$. For example, to any integrable real function $f(x)$ we can associate the linear functional $T_f : \mathcal{F} \to \mathbb{R}$ defined as

$$\phi(x) \to T_f(\phi) = \int_{-\infty}^{\infty} dx f(x) \phi(x).$$

(1.2)

The idea is that, knowing the value of the integral $T_f(\phi)$ on a large enough number of test functions $\phi$, we can reconstruct the original function $f$. Allowing for general linear functionals we extend the concept of function.

We can define operations on distributions even in cases where they are not well defined on functions. For example, we will be able to define the derivative of a discontinuous function when it is seen as a distribution. In the world of distributions, everything is infinitely differentiable and limits always commute with other operations. The distributions are also called generalized functions.
1.1 Test functions

In this section we discuss the space \( \mathcal{F} \) of test functions and its properties. A distribution will be a linear functional on \( \mathcal{F} \). For simplicity, we consider test functions of real variable. We want to choose test functions \( \phi \in \mathcal{F} \) that are as smooth as possible, in particular that are infinitely differentiable, \( \phi \in C^\infty(\mathbb{R}) \). Moreover, in order to guarantee the existence of the integral in (1.2) we require that \( \phi \) vanishes sufficiently rapidly at infinity. There are two canonical choices for \( \mathcal{F} \), the space of bump functions \( \mathcal{D}(\mathbb{R}) \) and the space of rapidly decreasing functions \( \mathcal{S}(\mathbb{R}) \), which we now discuss.

1.1.1 The space \( \mathcal{D}(\mathbb{R}) \)

The support of a function of real variable is the closure of the set of points where the function is non-zero

\[
\text{supp}(f) = \{ x \in \mathbb{R} | f(x) \neq 0 \} \tag{1.3}
\]

A function with compact support is a function whose support is compact (a closed and bounded subset of \( \mathbb{R} \)).

We call \( \mathcal{D}(\mathbb{R}) \) the space of real functions of real variable that are infinitely differentiable and with compact support,

\[
\mathcal{D}(\mathbb{R}) = \{ \phi(x) | \phi \in C^\infty(\mathbb{R}) \text{ with compact support} \} \tag{1.4}
\]

\( \mathcal{D}(\mathbb{R}) \) is obviously a vector space. It is also a topological space, but the notion of limit that is useful for the theory of distribution is not induced by any norm or metric. For our purposes, it is enough to define the notion of convergence of a sequence in the topology of \( \mathcal{D}(\mathbb{R}) \). We say that the sequence \( \{ \phi_n \in \mathcal{D}(\mathbb{R}) \} \) converges to the function \( \phi \in \mathcal{D}(\mathbb{R}) \) if the following conditions are satisfied,

- it exists a bounded interval \( \hat{I} \subset \mathbb{R} \) which contains the supports of all the functions \( \phi_n \) and \( \phi \). In other words, it exists a real number \( R \) such that
  \[
  \phi_n(x) = 0 \quad \text{and} \quad \phi(x) = 0 \quad \forall |x| \geq R; \tag{1.5}
  \]
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- the sequence of the $p$-th derivatives of $\phi_n(x)$ converges uniformly to the $p$-th derivative of $\phi(x)$

$$\sup_{x \in \mathbb{R}} \left| \frac{d^p}{dx^p} \phi_n(x) - \frac{d^p}{dx^p} \phi(x) \right| \xrightarrow{n \to \infty} 0 \quad \forall \ p = 0, 1, \ldots$$ (1.6)

In particular, $\phi_n(x)$ converges uniformly to $\phi(x)$.

The generalization of the space $\mathcal{D}(\mathbb{R})$ to the case of complex-valued functions or of functions defined in $\mathbb{R}^n$ is straightforward.

**Example 1.1.** The function

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$ (1.7)

belongs to $\mathcal{D}(\mathbb{R})$ since it is zero outside the interval $[-1, 1]$, infinitely differentiable in $(-1, 1)$ and with vanishing derivatives of all orders in $x = \pm 1$. Notice that the function is $C^\infty(\mathbb{R})$ but not analytic. The Taylor series in $x = \pm 1$ is identically zero since all derivatives are zero, but the function is not zero in the neighborhood of $x = \pm 1$. All the functions in $\mathcal{D}(\mathbb{R})$ are necessarily non analytic.

1.1.2 The space $\mathcal{S}(\mathbb{R})$

A function $f$ of real variable is called a rapidly decreasing function, or a Schwarz function, if it is infinitely differentiable on $\mathbb{R}$, $f \in C^\infty(\mathbb{R})$, and

$$x^p \frac{d^q}{dx^q} f(x) \xrightarrow{x \to \pm \infty} 0 \quad \forall \ p, q = 0, 1, \ldots$$ (1.8)

In other words, $f$ is infinitely differentiable and $f$ and all its derivatives vanish at infinity more rapidly than any polynomial.

The vector space of rapidly decreasing functions is denoted

$$\mathcal{S}(\mathbb{R}) = \{ \phi(x) \mid \phi \in C^\infty(\mathbb{R}) \text{ and rapidly decreasing} \}.$$ (1.9)

We define a notion of convergence in $\mathcal{S}(\mathbb{R})$ as follows. We say that the sequence of functions $\{\phi_n \in \mathcal{S}(\mathbb{R})\}$ converges to the function $\phi \in \mathcal{S}(\mathbb{R})$ if $x^p \frac{d^q}{dx^q} \phi_n(x)$ converges uniformly to $x^p \frac{d^q}{dx^q} \phi(x)$ for all $p$ and $q$

$$\sup_{x \in \mathbb{R}} \left| x^p \frac{d^q}{dx^q} \phi_n(x) - x^p \frac{d^q}{dx^q} \phi(x) \right| \xrightarrow{n \to \infty} 0 \quad \forall \ p, q = 0, 1, \ldots$$ (1.10)
Example 1.2. The functions

\[ \phi(x) = P(x)e^{-x^2}, \quad \phi(x) = \sin x e^{-x^2}, \]  

(1.11)

where \( P(x) \) is an arbitrary polynomial, are elements of \( \mathcal{S}(\mathbb{R}) \). On the other hand, the functions

\[ \phi(x) = \frac{1}{1 + x^2}, \quad \phi(x) = e^{-|x|} \]  

(1.12)

are not, for the first vanishes as an inverse polynomial at infinity and the second, while vanishing faster than any polynomial at infinity, is not \( C^\infty(\mathbb{R}) \).

Example 1.3. Since every function in \( \mathcal{S}(\mathbb{R}) \) is square integrable, \( \mathcal{S}(\mathbb{R}) \) is a vector subspace of \( L^2(\mathbb{R}) \). \( \mathcal{S}(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \). In fact, any element of \( L^2(\mathbb{R}) \) is the limit of a sequence of elements of \( \mathcal{S}(\mathbb{R}) \). To see this, it is enough to expand \( f \in L^2(\mathbb{R}) \) in the Hermite basis,

\[ f = \sum_{n=0}^{\infty} c_n u_n, \]

where \( u_n(x) \sim H_n(x)e^{-x^2} \) are obviously rapidly decreasing functions.

1.2 Distributions

We call distribution a functional \( T : \mathcal{F} \rightarrow \mathbb{C} \)

\[ \phi(x) \rightarrow T(\phi). \]  

(1.13)

where \( \mathcal{F} = D(\mathbb{R}) \) or \( \mathcal{F} = \mathcal{S}(\mathbb{R}) \), with the properties

- \( T \) is linear:

\[ T(\lambda_1 \phi_1 + \lambda_2 \phi_2) = \lambda_1 T(\phi_1) + \lambda_2 T(\phi_2), \]  

(1.14)

for all \( \lambda_1, \lambda_2 \in \mathbb{C} \) and \( \phi_1, \phi_2 \in \mathcal{F} \);

- \( T \) is continuous in the following sense: if the sequence \( \phi_n(x) \) converges to \( \phi(x) \) in \( \mathcal{F} \), \( \lim_{n \rightarrow \infty} \phi_n(x) = \phi(x) \), then

\[ \lim_{n \rightarrow \infty} T(\phi_n) = T(\phi). \]  

(1.15)
1.2. DISTRIBUTIONS

It is also customary to use the notation

\[ (T, \phi) \equiv T(\phi) \quad (1.16) \]

for the action of a distribution on a test function.

In the theory of distributions, the functions \( \phi \) in \( \mathcal{D}(\mathbb{R}) \) and \( \mathcal{S}(\mathbb{R}) \) are called test functions. The functionals \( T \) acting on \( \mathcal{F} = \mathcal{D}(\mathbb{R}) \) are simply called distributions and form a space denoted \( \mathcal{D}'(\mathbb{R}) \). The functionals \( T \) acting on \( \mathcal{F} = \mathcal{S}(\mathbb{R}) \) are called tempered distributions and form a space denoted \( \mathcal{S}'(\mathbb{R}) \).

We say that two distributions, \( T_1 \) et \( T_2 \), are equal if they act in the same way on all test functions

\[ T_1(\phi) = T_2(\phi) \quad \forall \phi \in \mathcal{D}(\mathbb{R}) (\mathcal{S}(\mathbb{R})) \quad (1.17) \]

The spaces of distributions \( \mathcal{D}'(\mathbb{R}) \) et \( \mathcal{S}'(\mathbb{R}) \) are vector spaces if we define sum and multiplication by scalars by

\[ (T_1 + T_2)(\phi) = T_1(\phi) + T_2(\phi) , \quad (\lambda T)(\phi) = \lambda T(\phi) , \quad (1.18) \]

for all \( \phi \in \mathcal{D}'(\mathbb{R}) (\mathcal{S}'(\mathbb{R})) \) and all complex numbers \( \lambda \).

The functions with compact support are a subset of the rapidly decreasing functions

\[ \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) . \quad (1.20) \]

This implies the opposite inclusion for the space of distributions

\[ \mathcal{S}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R}) . \quad (1.21) \]

since every functional that is defined on all the functions in \( \mathcal{S}(\mathbb{R}) \) is a fortiori defined on the functions in \( \mathcal{D}(\mathbb{R}) \).

It is straightforward to extend the previous definitions to the case of multi-variable distributions.

In the following we discuss the case of distributions in \( \mathcal{D}'(\mathbb{R}) \), since all properties we will introduce in this chapter are also valid for tempered distributions. More sophisticated operations, like the Fourier transform, can be only defined for tempered distributions.
1.2.1 Regular distributions

We say that a function \( f(x) \) is \textit{locally integrable}, and we write \( f \in L^1_{\text{loc}}(\mathbb{R}) \), if its integral on any compact subset \( K \) of \( \mathbb{R} \) is finite

\[
\int_K |f(x)| \, dx < \infty. \tag{1.22}
\]

Notice that we are not requiring any nice behavior at infinity. A function like \( f(x) = e^{x^2} \) is not an element \( L^1(\mathbb{R}) \) but it is in \( L^1_{\text{loc}}(\mathbb{R}) \). To any locally integrable function \( f \) we associate the linear functional

\[
T_f(\phi) = (f, \phi) = \int_{-\infty}^{\infty} dx f^*(x) \phi(x). \tag{1.23}
\]

\( T_f \) is a distribution \( T_f \in \mathcal{D}'(\mathbb{R}) \).

\textbf{Proof.} In order to see that \( T_f \) is a distribution we need to check that its value on a test function is well defined and that \( T_f \) is linear and continuous. Call \( K_\phi \) the support of \( \phi \in \mathcal{D}(\mathbb{R}) \). The integral in (1.23) restricts to \( \int_{K_\phi} dx f^*(x) \phi(x) \), which is finite since \( \phi \) is \( C^\infty(\mathbb{R}) \) and \( f \) is integrable on any compact. \( T_f \) is clearly linear in \( \phi \). To see that it is continuous take a sequence of test functions \( \phi_n \) converging to \( \phi \) in the topology of \( \mathcal{D}(\mathbb{R}) \). Recall that this means that the supports of \( \phi_n \) and \( \phi \) are contained in a compact set \( K \), and that \( \phi_n \) and all its derivatives converges uniformly to \( \phi \). We have

\[
|T_f(\phi) - T_f(\phi_n)| \leq \int_K dx |f(x)||\phi(x) - \phi_n(x)| \leq \sup_{x \in K} |\phi(x) - \phi_n(x)| \int_K |f|
\]

which converges to zero since \( f \) is locally integrable and \( \phi_n \) converges uniformly to \( \phi \): \( \lim_{n \to \infty} \sup_{x \in K} |\phi(x) - \phi_n(x)| = 0 \).

Distributions of the form \( T_f \) for a locally integrable function \( f \) are called \textit{regular distributions}. \( T_f \) is not in general a tempered distribution, because the integral (1.23) is not necessarily convergent at infinity. Since \( \phi \in \mathcal{S}(\mathbb{R}) \) vanishes at infinity more rapidly than any polynomials, \( T_f \) will be a tempered distribution if \( f \) has an \textit{algebraic growth}, which means that \( |f(x)| < c|x|^k \) for some \( k \) and \( c \) for large enough \( |x| \).

We can use definition (1.23) to \textit{identify} the space of functions \( L^1_{\text{loc}}(\mathbb{R}) \) with a subset of the space of distributions \( \mathcal{D}(\mathbb{R}) \). In this sense, distributions generalize functions. When no confusion can arise we will make use of the identification \( f \sim T_f \) and write \( f \) instead of \( T_f \).
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The presence of the complex conjugate of \( f \) in the definition (1.23) can be startling but it is dictated by convenience. In particular, with this choice, \( T_f(\phi) \) can be formally written as the scalar product \((f, \phi)\) of \( f \) with \( \phi \). This definition also explains the notation (1.16) which is often used for the action of a distribution on a test function.

**Example 1.4.** The function \( f(x) = \text{sign}(x) \) is not continuous, it is not differentiable and it is not integrable on \( \mathbb{R} \). It is however integrable on any finite interval \([a, b]\) and thus defines the distribution

\[
T_f(\phi) = \int_{-\infty}^{\infty} dx\, f(x)\, \phi(x) = -\int_{-\infty}^{0} dx\, \phi(x) + \int_{0}^{\infty} dx\, \phi(x).
\]

Since \( f \) has an algebraic growth, \( T_f \) is a tempered distribution. The previous expression is indeed finite for all \( \phi \in \mathcal{S}(\mathbb{R}) \).

**Example 1.5.** The Heaviside function

\[
\theta(x) = \begin{cases} 
1 & x \geq 0 \\
0 & x < 0
\end{cases},
\]

is similarly associated to the tempered distribution

\[
T_\theta(\phi) = \int_{-\infty}^{\infty} dx\, \theta(x)\, \phi(x) = \int_{0}^{\infty} dx\, \phi(x).
\]

**Example 1.6.** By identifying locally integrable functions with the associated distribution, we have the following inclusion: \( C^\infty(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R}) \). The elementary functions \( \sin(x), \cos(x), \log(x), P(x) \), where \( P(x) \) is a polynomial are tempered distributions. The function \( e^x \) is a distribution, but growing faster than a polynomial, is not tempered.

**Example 1.7.** A notable chain of inclusions is

\[
\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R}).
\]

In particular we will see that each space in this chain is dense in those containing it.
1.2.2 Singular distributions

Most distributions are not of the form \( (1.23) \) for any locally integrable function. They are called *singular* and are the truly new objects. We give various examples. In order to see that a functional is a distribution we always need to check that its value on a test function is well defined and it is linear and continuous. We will often leave as an exercise to the reader to verify the continuity. This can be done with the same method used for regular distributions.

**Example 1.8.** The Dirac delta function.

The most famous example of singular distribution is given by the *Dirac delta function* which is not a function, as the name would suggest, but the distribution defined by

\[
\delta_{x_0}(\phi) = \phi(x_0). \tag{1.26}
\]

In other words, \( \delta_{x_0} \) associates to the test function \( \phi(x) \) its value at the point \( x_0 \). It is obvious that the functional \( \delta_{x_0} \) is well defined and linear. It is also continuous since for a sequence \( \phi_n \to \phi \) in \( \mathcal{D}(\mathbb{R}) \)

\[
|\delta_{x_0}(\phi_n) - \delta_{x_0}(\phi)| = |\phi_n(x_0) - \phi(x_0)| \tag{1.27}
\]

converges to zero for \( n \to \infty \) because \( \phi_n \) converges uniformly, and therefore also pointwise, to \( \phi \). \( \delta_{x_0} \) is obviously well defined also for \( \phi \in \mathcal{S}(\mathbb{R}) \) and it is therefore a tempered distribution. In physics it is customary to use the following notation

\[
\delta_{x_0}(\phi) \equiv \int_{-\infty}^{\infty} \delta(x-x_0) \phi(x) \, dx, \tag{1.28}
\]

as if there were a *function* \( \delta(x-x_0) \) such that

\[
\int_{-\infty}^{\infty} \delta(x-x_0) \phi(x) \, dx = \phi(x_0). \tag{1.29}
\]

Unfortunately a function \( \delta(x-x_0) \) which satisfies the previous equation for all \( \phi \) does not exist. Even if we will use the notation \( \delta(x-x_0) \) we must keep in mind that \( \delta_{x_0} \) is not a function but a truly singular distribution. When \( x_0 = 0 \) we simply write \( \delta \) or \( \delta(x) \).
Example 1.9. The Principal Value.
The function $1/x$ is not locally integrable in the neighborhood of $x = 0$ and it cannot be considered as a distribution. We can however define a tempered distribution $T$ which is close enough to $1/x$ using the Principal Value

$$T(\phi) = \lim_{\epsilon \to 0} \left( \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} \, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x} \, dx \right).$$

The limit always exists since the sum of integrals can be written as

$$\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} \, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x} \, dx = \int_{\epsilon}^{\infty} \frac{\phi(x) - \phi(-x)}{x} \, dx,$$

and the integrand $\frac{\phi(x) - \phi(-x)}{x}$ is continuous in the neighborhood of $x = 0$ for all $C^\infty(\mathbb{R})$ functions. We can thus take the $\epsilon \to 0$ limit on the right hand side of (1.31) and write

$$T(\phi) = \int_{0}^{\infty} \frac{\phi(x) - \phi(-x)}{x} \, dx.$$

The distribution $T$ is also written with some abuse of notations as $P_1^x$ and can be viewed as a regularization of the function $1/x$ in the neighborhood of $x = 0$.

Example 1.10. Other regularizations of the function $1/x$ are given by the distributions

$$T_\pm(\phi) = \lim_{\epsilon \to 0} \int_{-\infty}^{\pm \infty} \frac{\phi(x)}{x \pm i\epsilon} \, dx.$$

The original singularity of the integrand is avoided by shifting the position of the singularity in the complex plane by a small imaginary part. The action of $T_\pm$ on a test function is often denoted with some abuse of notations as

$$T_\pm(\phi) = \int \frac{\phi(x)}{x \pm i0} \, dx.$$

and the distributions $T_\pm$ themselves as $1/(x \pm i0)$. The distributions $T_\pm$ are related to $P_1^x$ by the identities

$$\frac{1}{x \pm i0} = P_1^x \mp \pi i\delta(x).$$
We can indeed write
\[ T_\pm(\phi) = \lim_{\epsilon \to 0} \left( \int_{-\infty}^{+\infty} \frac{x}{x^2 + \epsilon^2} \phi(x) dx \mp i \int_{-\infty}^{+\infty} \frac{\epsilon}{x^2 + \epsilon^2} \phi(x) dx \right) \quad (1.36) \]

The first integral on the right hand side can be written as
\[ \int_{0}^{\infty} \frac{x^2}{x^2 + \epsilon^2} \left( \frac{\phi(x) - \phi(-x)}{x} \right) dx , \quad (1.37) \]

and thus converges for \( \epsilon \to 0 \) to the principal value \( (1.32) \). The second integral can be written, by the change of variables \( x = \epsilon y \), as
\[ \mp i \int_{-\infty}^{+\infty} \frac{\epsilon}{x^2 + \epsilon^2} \phi(x) dx = \mp i \int_{-\infty}^{+\infty} \frac{1}{y^2 + 1} \phi(\epsilon y) dy , \quad (1.38) \]

which converges for \( \epsilon \to 0 \) to \( \mp i\phi(0) \int_{\mathbb{R}} \frac{dy}{y^2 + 1} = \mp i\pi\phi(0) \). We then have
\[ \int \frac{\phi(x)}{x+i0} dx = P \int \frac{\phi}{x} - \pi i \int \delta(x) \phi(x) dx \quad (1.39) \]

which, being valid for all test functions \( \phi \), gives us the identity \( (1.35) \).

**Example 1.11.** The finite part.
The principal value for a function which behaves as \( 1/x^2 \) is not well defined.

We can modify the expression \( (1.32) \) as follows
\[ T(\phi) = \int_{0}^{\infty} \frac{\phi(x) + \phi(-x) - 2x\phi'(0)}{x^2} dx . \quad (1.40) \]

which is now well defined since the numerator is \( O(x^2) \) in the neighborhood of \( x = 0 \) by Taylor theorem. \( T \) in \( (1.40) \) defines a distribution, called the finite part and denoted as \( \text{fp}_{1/x^2} \), which regularizes the function \( 1/x^2 \) in the neighborhood of \( x = 0 \).

A distribution is only defined by its action on test functions and, even if we often denotes distributions as "functions" like in the case of \( \delta(x) \) or \( 1/(x+i0) \), we must remember that these are formal expressions and distributions are not defined pointwise. The concept of value of the distribution \( T \) at the point \( x \) is meaningless. Only for regular distributions \( T_f \) we can identify the distribution with the function \( f \) and define the value of \( T \) at a point \( x \) as
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the value \( f(x) \), if it exists. More generally, we can study the behavior of a distribution \( T \) on a subset \( I \) of \( \mathbb{R} \) by considering the action of \( T \) on test functions with support in \( I \). If, on all test functions with support in \( I \), the action of \( T \) is the same as the action of a regular distribution \( T_f \), we can say that \( T \) is equal to \( f \) in the interval \( I \) and we can talk about the values of \( T \) at points of \( I \). For example, \( \delta(x) \) is identified with the function zero and \( P1/x \) and \( 1/(x \pm i0) \) with the function \( 1/x \) on all intervals that do not contain \( x = 0 \). We can define the support of \( T \) as the closure of the complement of the biggest subset of \( \mathbb{R} \) where it can be identified with the zero function. The support of \( \delta_{x_0} \) is the point \( x_0 \).

**Example 1.12.** Compactly supported distributions.

In general, the action of a distribution \( T \in \mathcal{D}'(\mathbb{R}) \) is only defined on \( C^\infty \) functions with compact support. However if \( T \) itself has compact support, it effectively truncate the domain of definition of any functions is acting on. A distribution \( T \) with compact support can act on any \( C^\infty \) function. In practice, if \( K = \text{Supp}(T) \) and \( \phi \in C^\infty(\mathbb{R}) \), we can always find \( \tilde{\phi} \in \mathcal{D}(\mathbb{R}) \) which coincides with \( \phi \) in \( K \) and define \( T(\phi) \equiv T(\tilde{\phi}) \).

1.3 Limits of distributions

We can define a notion of convergence for distributions. We say that the sequence \( \{ T_n \} \in \mathcal{D}'(\mathbb{R}) \) converges to \( T \) in the sense of distributions and we write \( \lim_{n \to \infty} T_n = T \), if, for all test functions \( \phi \in \mathcal{D}(\mathbb{R}) \), we have

\[
\lim_{n \to \infty} T_n(\phi) = T(\phi) .
\]  

(1.41)

When the distributions \( T_n \) are regular and associated with locally integrable functions \( f_n \), the distributional limit of \( T_n \) is not necessarily the same of the pointwise (or uniform or \( L^p \)) limit of the \( f_n \). The limit in \( \mathcal{D}'(\mathbb{R}) \) is often called limit in weak sense, since it is defined by considering the action of \( T_n \) on test functions.

**Example 1.13.** The sequence of locally integrable functions \( f_n(x) = e^{inx} \) for \( n = 1, 2, 3, \ldots \) converges pointwise to 1 for \( x = 2\pi k \) avec \( k \in \mathbb{Z} \) and is not
convergent for all other \( x \). On the other hand, the sequence of distributions \( T_{f_n} \) converge to 0 as we see with an integration by parts

\[
(T_{f_n}, \phi) = \int_{-\infty}^{+\infty} e^{-inx} \phi(x) dx = \frac{1}{in} \int_{-\infty}^{+\infty} e^{-inx} \phi'(x) dx \xrightarrow{n \to \infty} 0. \quad (1.42)
\]

**Example 1.14.** The sequence of locally integrable functions

\[
f_k(x) = \begin{cases} 
  k & \frac{1}{k} < x < \frac{2}{k} \\
  0 & \text{elsewhere}
\end{cases} \quad (k \geq 1) \quad (1.43)
\]

converges pointwise to 0. The associated distributions converge instead to the Dirac delta function,

\[
\lim_{k \to \infty} (T_{f_k}, \phi) = \lim_{k \to \infty} k \int_{\frac{1}{k}}^{\frac{2}{k}} dx \phi(x) = \lim_{k \to \infty} \phi(x_0) = \phi(0) = (\delta_0, \phi) \quad (1.44)
\]

where \( x_0 \) is a point of the interval \( \left[ \frac{1}{k}, \frac{2}{k} \right] \) and we used the mean value theorem for the \( C^\infty \) function \( \phi \).

**Example 1.15.** The Dirac delta function \( \delta(x) \) can be obtained as a limit of any of the following sequences of functions

\[
f_n(x) = \left\{ \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}, \frac{\sin nx}{\pi x}, \frac{1}{n\pi} \left( \frac{\sin nx}{x} \right)^2, \frac{1}{n\pi x^2 + 1/n^2} \right\} \quad (1.45)
\]

All these functions are obtained by rescaling: \( f_n(x) = nf(nx) \), where \( f(x) \) is a locally integrable function of unit area, \( \int_{-\infty}^{\infty} f(x) dx = 1 \). Given a test function \( \phi \), we have

\[
\int_{-\infty}^{\infty} f_n(x)\phi(x) dx \xrightarrow{y/n} \int_{-\infty}^{\infty} f(y)\phi(y/n) dy \xrightarrow{n \to \infty} \phi(0) \int_{-\infty}^{\infty} f(y) dy = \phi(0).
\]

if \( f \) is sufficiently regular. It follows that \( \lim f_n(x) = \delta(x) \) in distributional sense. Notice that the pointwise limit of all the functions \( f_n(x) \) is 0 for \( x \neq 0 \) and \( +\infty \) for \( x = 0 \). It is sometime said that the Dirac delta function \( \delta(x) \) is vanishing for \( x \neq 0 \) and goes to infinity for \( x = 0 \) and it is not uncommon on physics textbook to find the writing

\[
\delta(x) = \begin{cases} 
  0 & x \neq 0 \\
  \infty & x = 0
\end{cases} \quad (1.46)
\]
We stress again that the concept of pointwise value of a distribution $f(x)$ is not well defined. In particular the value of $\delta$ in $x = 0$ makes no sense. What it is true is that $\delta(x)$ is the limit of functions $f_n$ which become more and more supported around $x = 0$ and whose value in the origin increases when $n$ becomes large.

The previous example shows that the weak limit of locally integrable functions can be a singular distribution. In fact, one can show that all singular distributions can be obtained as a limit of locally integrable functions. One can show indeed that $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{D}'(\mathbb{R})$.

1.4 Operations on distributions

Many operations that are defined for functions can be extended to distributions. The idea is to let the operation act on the test function. One usually defines the operation for regular distributions, move the operation on the test function and extend the definition to the singular distributions. Since test functions are $C^\infty(\mathbb{R})$, it follows, for example, that all distributions are differentiable.

1.4.1 Change of variables

Distributions are not defined point-wise but we can nevertheless define the concept of change of variables. This is easy to do for a regular distribution $T_f$, associated with the locally integrable function $f(x)$. We want to consider now the function $g(x) = f(u(x))$, or in short $g = f \circ u$, where $u(x) \in C^\infty(\mathbb{R})$ and is one-to-one map of $\mathbb{R}$ to itself. We have

$$T_g(\phi) = \int_{-\infty}^{+\infty} dx f(u(x)) \phi(x) \quad y = u(x) = \int_{-\infty}^{+\infty} dy \frac{f(y)}{|u'(u^{-1}(y))|} \phi(u^{-1}(y)), \quad (1.47)$$

or

$$T_{f \circ u}(\phi) = T_f \left( \frac{\phi \circ u^{-1}}{|u' \circ u^{-1}|} \right). \quad (1.48)$$

What we did was to move the operation (the change of variable) from $T$ to the test function. Now that the operation is only acting on the test function
we can generalize the previous formula and define, for any distribution $T$, the distribution $T \circ u$, defined as

$$T \circ u(\phi) := T(\phi \circ u^{-1}/|u' \circ u^{-1}|).$$

(1.49)

The definition makes sense since $\phi \circ u^{-1}/|u' \circ u^{-1}|$ is a test function, being $C^\infty(\mathbb{R})$ and with compact support. One can easily check that the functional $T \circ u$ is continuous. The above definition may look complicated and it is better to demonstrate it by examples.

**Example 1.16. Translation**

Given $x_0 \in \mathbb{R}$ and the distribution $\delta_{x_0}$ we want to find the distribution translated by $x \rightarrow T_a(x) = x + a$. Using (1.49) we have

$$\delta_{x_0} \circ T_a(\phi) = \delta_{x_0} (\phi \circ T_a^{-1}) = \int_{\infty}^{\infty} \delta(x - x_0) \phi(x - a) \, dx = \phi(x_0 - a)$$

(1.50)

and we find, not unexpectedly, that the translation $\delta_{x_0} \circ T_a$ is $\delta_{x_0 - a}$.

**Example 1.17. Change of scale**

Given $\lambda \in \mathbb{R}$ and the distribution $\delta$ we want to find the distribution obtained by a dilatation $x \rightarrow D_\lambda(x) = \lambda x$. Using (1.49) we have

$$\delta \circ D_\lambda(\phi) = \delta_x (\phi \circ D_\lambda^{-1}) = \int_{\infty}^{\infty} \delta(x) \phi(x/\lambda) \frac{1}{|\lambda|} \, dx = \phi(0)/|\lambda|$$

(1.51)

and we find that $\delta \circ D_\lambda = \delta/|\lambda|$. With the usual abuse of notation we can write the more transparent expression $\delta(\lambda x) = \delta(x)/|\lambda|$.

**Example 1.18. Change of variables in the delta function**

For a more general change of variables $x \rightarrow u(x)$, with $u$ monotonic and with a single zero in $x = x_0$, we would obtain from (1.49)

$$\delta(u(x)) = \frac{\delta(u^{-1}(0))}{|u'(u^{-1}(0))|} = \frac{\delta(x_0)}{|u'(x_0)|}$$

(1.52)

Notice that that $\delta(u(x))$ evaluates the test function in the zeros of $u(x)$, as expected, but it also multiplies it by a constant. Since the delta function is vanishing outside $x = 0$ we can generalize the construction by using functions
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\( u(x) \) which are not necessarily monotonic but have a finite number of zeros \( x_i \) with \( u'(x_i) \neq 0 \). In this case we have

\[
\delta(u(x)) = \sum_i \frac{\delta(x_i)}{|u'(x_i)|}
\]

(1.53)

For example, \( \delta(x^2 - 1) = \delta(x-1)^2 + \delta(x+1)^2 \).

1.4.2 Multiplication by a \( C^\infty \) function

We define the product of a distribution \( T \) by a function \( g \in C^\infty \) as

\[
(gT)(\phi) := T(g\phi).
\]

(1.54)

The definition makes sense since \( g\phi \) is a test function, being \( C^\infty(\mathbb{R}) \) and with compact support. Notice that it is important to have \( g \in C^\infty \) in order to have \( g\phi \in \mathcal{D}(\mathbb{R}) \). One can easily check that the functional \( gT \) is continuous.

The definition is motivated by the case of a regular distribution \( T_f \) where indeed

\[
(T_g f, \phi) = \int_{-\infty}^{\infty} dx \, g(x) f(x) \phi(x) = (T_f, g\phi).
\]

(1.55)

Example 1.19. We want to show that

\[
xP_{\frac{1}{x}} = 1
\]

(1.56)

in \( \mathcal{D}'(\mathbb{R}) \). Notice that 1 is an element of \( \mathcal{D}'(\mathbb{R}) \), being locally integrable. Denoting \( T = P_{\frac{1}{x}} \) we have indeed

\[
xT(\phi) = T(\phi) = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} dx \, \frac{x \phi(x)}{x} = \int_{-\infty}^{\infty} dx \phi(x) = T_1(\phi)
\]

(1.57)

since the integral in principal value of the \( C^\infty \) function \( \phi \) is the same as the ordinary integral. Similarly we prove

\[
x \frac{1}{x + i0} = 1.
\]

(1.58)

Example 1.20. The general solution of the algebraic equation \( xT = 1 \) for \( T \) in \( \mathcal{D}'(\mathbb{R}) \) is

\[
T = P_{\frac{1}{x}} + c\delta(x)
\]

(1.59)
where \( c \) is an arbitrary constant. In fact \( x\delta = 0 \) identically in \( \mathcal{D}'(\mathbb{R}) \) since
\[
 x\delta(\phi) = \delta(x\phi) = \int_{-\infty}^{\infty} \delta(x) x\phi(x) = 0 \tag{1.60}
\]
For \( c = \mp i\pi \) we have \( T = \frac{1}{ix} \), as Example 1.10 shows. Notice that the inverse in the space of distributions \( \mathcal{D}'(\mathbb{R}) \) is not unique.

### 1.4.3 Derivative of a distribution

Given a locally integrable function \( f \) whose derivative \( f' \) exists and it is locally integrable, we can consider the distribution \( T_f \in \mathcal{D}'(\mathbb{R}) \) and write
\[
 T_f(\phi) = \int_{-\infty}^{+\infty} \left[ f(x) \phi(x) - \int_{-\infty}^{x} f(y) \phi'(y) dy \right] dx \]
\[
 = - \int_{-\infty}^{+\infty} \left[ f(x) \phi'(x) - \int_{-\infty}^{x} f(y) \phi''(y) dy \right] dx \tag{1.61}
\]
where we moved the operation (the derivative) on the test function by integrating by parts. It is important to notice that we can do that since \( \phi \) is \( C^\infty \) and the boundary terms vanish since \( \phi \in \mathcal{D}(\mathbb{R}) \) has compact support.

This computation suggests to define the derivative \( T' \) for an arbitrary distribution \( T \in \mathcal{D}'(\mathbb{R}) \) as
\[
 T'(\phi) := -(T, \phi') \tag{1.62}
\]
The previous definition makes sense since \( \phi' \) is still a test function. It is easy to check that \( T' \) is continuous.

Since test functions are \( C^\infty \), we can iterate the operation of derivation an infinite number of times: the \( n \)-th derivative of \( T \) will be given by
\[
 T^{(n)}(\phi) = (-1)^n T(\phi^{(n)}) \tag{1.63}
\]
where \( \phi^{(n)} \) denotes the \( n \)-th derivative of the test function \( \phi \). Distributions are infinitely differentiable!

The derivative of distributions has properties similar to those of functions. If \( T_1 \) and \( T_2 \) are two distributions we have
\[
 (\lambda_1 T_1 + \lambda_2 T_2)' = \lambda_1 T_1' + \lambda_2 T_2' \quad \forall \lambda_1, \lambda_2 \in \mathbb{C}, \tag{1.64}
\]
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\[ (T(x - x_0))' = T'(x - x_0) \quad \forall x_0 \in \mathbb{R}, \]  \hspace{1cm} (1.65)

\[ (T(ax))' = aT'(ax). \]  \hspace{1cm} (1.66)

All locally integrable functions \( f \), even if not differentiable in ordinary sense in one or more points, have a derivative \( T'_f \) when considered as distributions.

**Example 1.21.** The derivative of the distribution associated with the Heaviside function \( \theta(x) \) of Example 1.5 is

\[ T'_\theta(\phi) = -T_\theta(\phi') = - \int_{-\infty}^{\infty} dx \theta(x)\phi'(x) = - \int_{0}^{\infty} dx \phi'(x) = \phi(0). \]  \hspace{1cm} (1.67)

We thus find \( T'_\theta = \delta \). The standard derivative of the Heaviside function does not exist, but the distributional derivative does and it is given by the delta function. As expected, the distributional derivative of \( \theta(x) \) vanishes outside \( x = 0 \). The discontinuity (jump) in \( x = 0 \) gives rise to a delta function. With some abuse of language we will also write \( \theta' \) instead of \( T'_\theta \).

More generally, the distributional derivative allows to make sense of the derivative of discontinuous functions. Given a function \( f \) which is continuous and differentiable with continuous derivative everywhere except at the point \( x_0 \) where it has a jump discontinuity, we define the element \( f' \in L^1_{\text{loc}}(\mathbb{R}) \) as the (discontinuous) locally integrable function that coincides with the ordinary derivatives of \( f(x) \) in \( \mathbb{R} - \{x_0\} \)\(^1\). We have

\[ T'_f(\phi) = - \int_{-\infty}^{\infty} dx f(x)\phi'(x) =
\]

\[ = - f(x)\phi(x)|_{-\infty}^{x_0} + \int_{-\infty}^{x_0} dx f'(x)\phi(x) - f(x)\phi(x)|_{x_0}^{\infty} + \int_{x_0}^{\infty} dx f'(x)\phi(x) \]

\[ = \int_{-\infty}^{\infty} dx f'(x)\phi(x) + [f(x_0^+) - f(x_0^-)]\phi(x_0) \]  \hspace{1cm} (1.68)

where we have integrated by parts in the intervals \([-\infty, x_0]\) and \([x_0, +\infty]\).

By defining the discontinuity \( \text{disc}(f, x_0) = f(x_0^+) - f(x_0^-) \), we can write

\[ T'_f = T_{f'} + \text{disc}(f, x_0)\delta_{x_0}. \]  \hspace{1cm} (1.69)

\(^1\)We can define the value of \( f' \) in \( x_0 \) arbitrarily: the value in \( x_0 \) does not matter for integration and functions in \( L^1 \) which differs at a point are identified in the \( L^1 \).
Example 1.22. The function $\ln |x|$ is locally integrable in the neighborhood of $x = 0$ and thus defines a regular distribution. As a function it is not differentiable in ordinary sense in $x = 0$. In $\mathcal{D}'(\mathbb{R})$ we have the identity

$$\frac{d \ln |x|}{dx} = P \frac{1}{x}$$

as the following computation shows

$$T_{\ln |x|}(\phi) = -\int_{-\infty}^{+\infty} dx \ln |x| \phi'(x) = -\lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} dx \ln |x| \phi'(x) =$$

$$\lim_{\epsilon \to 0} \left[ \ln \epsilon \frac{\phi(\epsilon) - \phi(-\epsilon)}{\epsilon} + \int_{|x| \geq \epsilon} dx \frac{1}{x} \phi(x) \right] = P \int_{-\infty}^{\infty} dx \frac{1}{x} \phi(x).$$

The boundary terms vanish since

$$\lim_{\epsilon \to 0} (\epsilon \ln \epsilon) \frac{\phi(\epsilon) - \phi(-\epsilon)}{\epsilon} \sim 0(2\phi'(0)) = 0.$$  \hfill (1.71)

Similarly we can show that in $\mathcal{D}'(\mathbb{R})$

$$\frac{d^2 \ln |x|}{dx^2} = -\text{p} \frac{1}{x^2}.$$ \hfill (1.72)

1.4.4 Convolution of distributions

Distributions are not defined pointwise and therefore we cannot multiply two distributions. For example, the product $\delta(x)^2$ does not make any sense: formally we would write

$$\int_{-\infty}^{\infty} \delta(x)\delta(x) \phi(x) dx = \delta(0)\phi(0) = \infty.$$ \hfill (1.73)

The natural product on distributions is the convolution.

The convolution of two functions $f$ and $g$ is defined as the integral

$$(f * g)(x) = \int_{-\infty}^{\infty} dx' f(x')g(x - x').$$ \hfill (1.74)

One can show that if $f, g \in L^1(\mathbb{R})$ then $f * g \in L^1(\mathbb{R})$. More generally, if $f$ and $g$ are locally integrable and one of them has compact support, then $f * g$ is again locally integrable. It is easy to see that the convolution of functions
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satisfies the following properties. Given three functions \( f_1, f_2 \) et \( f_3 \), two of which at least with compact support, and two complex numbers \( \lambda, \mu \) we have

\[
f_1 \ast f_2 = f_2 \ast f_1, \tag{1.75}
\]

\[
(f_1 \ast f_2) \ast f_3 = f_1 \ast (f_2 \ast f_3), \tag{1.76}
\]

\[
f_1 \ast (\lambda f_2 + \mu f_3) = \lambda (f_1 \ast f_2) + \mu (f_1 \ast f_3), \tag{1.77}
\]

\[
(f_1 \ast f_2)' = f_1' \ast f_2 = f_1 \ast f_2', \tag{1.78}
\]

the last equation being a consequence of the obvious identity

\[
\int_{-\infty}^{\infty} dx' f'(x') g(x - x') = \int_{-\infty}^{\infty} dx' f(x - x') g(x'), \tag{1.79}
\]

obtained by changing variables \( x' \rightarrow x - x' \) in the integral.

As usual, we try to find a convenient definition for the convolution of two distributions by considering first the case of regular distributions. Consider then two locally integrable functions \( f \) and \( g \) with at least one of them with compact support. The regular distribution \( T_{f \ast g} \) acts as

\[
T_{f \ast g}(\phi) = \int_{-\infty}^{\infty} dx \left( \int_{-\infty}^{\infty} dy f(y) g(x - y) \right) \phi(x) \]

\[
= \int_{-\infty}^{\infty} dy f(y) \left( \int_{-\infty}^{\infty} dx g(x) \phi(x + y) \right). \tag{1.80}
\]

We can interpret the last expression as follows. We first act with the distribution \( T_g \) of variable \( x \) on the function \( \phi(x + y) \) considering \( y \) as a parameter and then we act on the result, which is a function of \( y \), with the distribution \( T_f \). Using the notation \((1.16)\) and abusing it we can write

\[
\left( T_f(y), T_g(x), \phi(x + y) \right), \tag{1.81}
\]

where we indicated explicitly the variables the distributions are acting on. Notice that \((T_g(x), \phi(x + y))\) is a \( C^\infty \) function of \( y \), since \( \phi \) is \( C^\infty \). If \( g \) has compact support, also \((T_g(x), \phi(x + y))\) is and it is a test function in \( \mathcal{D}(\mathbb{R}) \) to which we can apply \( T_f \). If \( g \) is not compactly supported, \((T_g(x), \phi(x + y))\) is not a test function and the previous expression does not make sense. However, we assumed that if \( g \) is not compactly supported, \( f \) is. \( T_f \) can
then act on any $C^\infty$ function because the support of $T_f$ effectively reduce the integral to a compact set (see Example 1.12).

We are now ready to generalize the construction to distributions. Recall from Section 1.2.2 and Example 1.12 that the concept of support of a distribution is well defined and that distributions with compact support can act on any $C^\infty$ function. Given two distributions $T$ and $S$, one of them with compact support, we define the convolution $T \ast S \in D(\mathbb{R})$ as

$$T \ast S(\phi) = (T(y), (S(x), \phi(x + y))).$$  \hfill (1.82)

One can show that $(S(x), \phi(x + y))$ is a $C^\infty$ function of $y$ and that it is compactly supported if $S$ is. If $S$ has compact support, $(S(x), \phi(x+y))$ is then a test function and the previous definition makes sense for any distribution $T$. If $S$ is not compactly supported $T$ will be and its action on $(S(x), \phi(x + y))$ is well defined even if the function is only $C^\infty$.

One can show that the convolution for distributions satisfies the properties (1.75)-(1.78).

**Example 1.23.** The delta function is an identity element for the convolution product

$$\delta \ast T = T \ast \delta = T.$$  \hfill (1.83)

With an obvious abuse of notation we have indeed

$$(\delta \ast T, \phi) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \delta(x)T(y)\phi(x + y) = \int_{-\infty}^{\infty} dy T(y)\phi(y) = (T, \phi).$$

### 1.4.5 Fourier Transform of Distributions

We can actually extend the Fourier Transform to tempered distributions. As usual, we first look at the case of regular distributions. The regular distribution $\hat{T}_g$ corresponding to the Fourier Transform of a function $g \in S(\mathbb{R})$ acts on an element $\varphi \in S(\mathbb{R})$ as

$$(\hat{T}_g, \varphi) = \int_{-\infty}^{\infty} dp \hat{g}(p) \varphi(p).$$  \hfill (1.84)

By inserting the explicit form of the Fourier Transform of $g$, and doing formal manipulations, one can also write

$$(\hat{T}_g, \varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ g(x) \int_{-\infty}^{\infty} dp \ \varphi(p) \ e^{-ipx} ,$$  \hfill (1.85)
which can be interpreted as the action of the regular distribution $T_g$ on the Fourier Transform of $\varphi$. In formulae,
\[
(T_g, \varphi) = \int_{-\infty}^{\infty} dp \; \hat{g}(p) \; \varphi(p) = \int_{-\infty}^{\infty} dx \; g(x) \; \hat{\varphi}(x) = (T_g, \hat{\varphi}). \tag{1.86}
\]
A generalization of (1.86) leads to the definition of the Fourier Transform of a generic distribution $T \in \mathcal{S}'(\mathbb{R})$
\[
\hat{T}(\varphi) = T(\hat{\varphi}) \tag{1.87}
\]
This definition extends the Fourier Transform from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. It is not hard to prove that the extension is a linear, bijective and bicontinuous operator from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. Notice that we cannot define the Fourier Transform for all distributions: the definition (1.87) would fail for $T \in \mathcal{D}'(\mathbb{R})$ since the Fourier Transform of $\varphi \in \mathcal{D}(\mathbb{R})$ is not necessarily in $\mathcal{D}(\mathbb{R})$.

**Example 1.24.** The Fourier Transform of the Dirac delta function is a constant: $\hat{\delta} = \frac{1}{\sqrt{2\pi}}$. Indeed
\[
(\hat{\delta}, \varphi) = (\delta, \hat{\varphi}) = \hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \varphi(x) = \left( \frac{1}{\sqrt{2\pi}}, \varphi \right). \tag{1.88}
\]
Notice that we could have obtained the same result by the formal manipulation
\[
\hat{\delta}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; \delta(x) \; e^{-ipx} = \frac{1}{\sqrt{2\pi}}. \tag{1.89}
\]
Consequently, the inverse theorem tells us that the Fourier Transform of the function $1$ is $\mathcal{F}(1)(\lambda) = \sqrt{2\pi} \delta(p)$. This can be interpreted as an integral representation of the delta function, namely
\[
\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \; e^{ipx}. \tag{1.90}
\]
This integral obviously does not exist in conventional sense, however it makes sense in distributional sense, namely when applied to a test function.

**Example 1.25.** Any locally integrable function with algebraic growth is in $\mathcal{S}'(\mathbb{R})$ and has a well defined Fourier Transform. Take for example a polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$. We find
\[
\mathcal{F}(P)(p) = \sum_{k=0}^{n} a_k i^k \partial_p^k \mathcal{F}(1)(p) = \sqrt{2\pi} \sum_{k=0}^{n} a_k i^k \delta^{(k)}(p), \tag{1.91}
\]
which is a distribution with support in the point \( p = 0 \). Vice-versa, the Fourier Transform of a linear combination of \( \delta(x) \) and its derivatives is a polynomial. One can show more generally that the Fourier Transform of any distribution with compact support is a \( C^\infty \) function with algebraic growth.

**Example 1.26.** Consider the Fourier Transform of the distribution \( \frac{1}{x+i0} \). We can easily compute it with the residue theorem

\[
\lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ipx}}{x+i\epsilon} = -\sqrt{2\pi i} \lim_{\epsilon \to 0} e^{-ip\theta(p)} = -\sqrt{2\pi i} \theta(p),
\]

where we close the contour in the upper (lower) half plane for \( p < 0 \) (\( p > 0 \)). The inverse Fourier Transform gives us a useful integral representation

\[
\frac{i}{x+i0} = \int_{0}^{\infty} e^{ipx} dp.
\]

It is interesting to observe that this formula can be obtained as a formal limit for \( \epsilon \to 0 \) of

\[
\frac{i}{x+i\epsilon} = \int_{0}^{\infty} e^{ipx-\epsilon p} dp,
\]

where now the integral converges in traditional sense for all \( \epsilon > 0 \).

Some of the properties of Fourier Transform of functions extend to the Fourier Transform of distributions. For instance

\[
\mathcal{F}(T^{(n)})(p) = (ip)^n \mathcal{F}T(p)
\]

\[
\mathcal{F}(x^n T)(p) = i^n \frac{dp^n}{dp^n} \mathcal{F}T(p)
\]

\[
\mathcal{F}(T_1 \ast T_2) = \mathcal{F}T_1 \mathcal{F}T_2
\]

The last property deserves some comments. First of all, we need to assume that at least one of the two distributions \( T_1 \) and \( T_2 \) has compact support, since, as we saw in Section 1.4.4, this a necessary condition for the convolution \( T_1 \ast T_2 \) to exist. Under this hypothesis, one can show that \( T_1 \ast T_2 \) is a tempered distribution, so that the left hand side of (1.97) is well defined. Moreover, as we already mentioned, the Fourier Transform of a tempered distribution with compact support is actually a \( C^\infty \) function with algebraic growth. It is only this property that allows to make sense of the right hand side of (1.97) as the product of a tempered distribution with a \( C^\infty \) function. Remember that the product of two distributions is generically not defined.
1.5 Exercises

Exercise 1.1. Show that the $k$-th derivative of the delta function acts as $(\delta^{(k)}, \phi) = (-1)^k \phi^{(k)}(0)$.

Exercise 1.2. Define a $\mathcal{D}(\mathbb{R}^2)$ distribution by $(f, \phi) = \int_0^\infty \int_0^\infty \phi(x, y) dxdy$. Compute $\partial^2 f/\partial x \partial y$.

Exercise 1.3. Show that $\frac{d^2 \ln |x|}{dx^2} = -f_0 \frac{1}{x^2}$ in $\mathcal{D}(\mathbb{R})$.

Exercise 1.4. Define the distribution $F = \log(x + i0)$ in $\mathcal{D}(\mathbb{R})$ by $(F, \phi) = \lim_{\epsilon \to 0} \int_{\infty}^{-\infty} \log(x + i\epsilon) \phi(x) dx$.
1) Show that $\log(x + i0) = \log |x| + i\pi \theta(-x)$. Hint: analyze log argument in the complex plane.
2) Show that $\frac{d}{dx} \log(x + i0) = 1/(x + i0)$
3) Check consistency of 1) and 2) with (1.35) and Example 1.22.

Exercise 1.5. Solve for $T$ the algebraic equation $P(x)T = Q(x)$ in $\mathcal{D}(\mathbb{R})$, where $P$ and $Q$ are real polynomials with distinct zeros.

Exercise 1.6. Solve for $T$ the differential equation $x^2 \frac{dT}{dx} = 1$ in $\mathcal{D}(\mathbb{R})$. Hint: solve it first for functions $T$ and add delta-function like ambiguities.

Exercise 1.7. Show that $T = \sum_{n=\infty}^{\infty} \delta(x - n)$ is a tempered distribution.
1) Expand formally the periodic distribution $T$ in Fourier series in $[0, 2\pi]$.
2) Show that the result in 1) is equivalent to the Poisson resummation formula valid for functions in $\mathcal{S}(\mathbb{R})$: $\sum_{k=-\infty}^{\infty} \hat{\phi}(k) = \sum_{n=-\infty}^{\infty} \hat{\phi}(n)$ where $\hat{\phi}(p) = \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i x}$ is the Fourier transform of $\phi$. 